(3) The locus of a point which lies at some constant distance from the curve on its inner normal must be such that it is also the locus of a point fixed on a bar of some simple linkage. In estimating the value of the word "simple" pivoted bars are preferred to slides and the total number should be as small as possible.

Condition (3) is needed to enable a cylinder to be ground accurately to the curve.

## CALCULUS.

**417. Proposed by H. S. UHLER, Yale University.**

To the degree of approximation indicated, show that

$$(\sqrt{-1})^{\sqrt{-1}} = 0.207,879,576,351.$$

**418. Proposed by B. F. FINKEL, Drury College.**

A rectangular tract of land is to be bought for the purpose of laying out a quarter-mile track with straightaway sides and semicircular ends. In addition a strip 35 yards wide along each straightaway is to be bought for grandstands, training quarters, etc. If the land costs $200 an acre, what will be the least possible cost of the land required?

GRANVILLE'S *Differential and Integral Calculus*, p. 116.

Is there anything wrong with this problem? Explain the contradiction involved in the solution.

## MECHANICS.

**334. Proposed by HORACE OLSON, Chicago, Illinois.**

A particle of elasticity $e$ is projected with a velocity $v$ at an angle $\phi$ with a plane inclined to the horizontal at an angle $\psi$; its plane of motion is perpendicular to the inclined plane. Show that after $2v \sin \phi/g(1 - e) \cos \psi$ seconds it will cease to rebound and will move along the plane with an initial velocity $v \cos \phi - 2v \sin \phi \tan \psi/(1 - e)$ and a uniform acceleration, $g \sin \psi$, down the plane.

(This problem is a generalization of problem 289 in Mechanics, which appeared in the March, 1914, issue of the MONTHLY.)

**335. Proposed by HAROLD T. DAVIS, Colorado Springs, Colorado.**

A heavy particle is projected upwards with a velocity $V$ in a medium resisting as the $n$th power of the velocity. Prove that the elevation of the particle when the velocity downwards is $V$ is equal to $LT$ where $L$ is the limiting velocity and $T$ is the time in which the particle falling from rest in the medium will acquire a velocity $V^2/L$.

## NUMBER THEORY.

**254. Proposed by HORACE OLSON, Chicago, Illinois.**

Find three integers, $x$, $y$, and $z$, such that $x^2 + y^2$, $x^2 + z^2$, $y^2 + z^2$, and $x^2 + y^2 + z^2$ are all perfect squares.

# SOLUTIONS OF PROBLEMS.

## ALGEBRA.

**456. Proposed by PAUL CAPRON, U. S. Naval Academy.**

If

$$S_{i,n} = \sum_{k=1}^{k=n-i+1} \frac{(i + k - 1)!}{(k - 1)!},$$

show that $S_{i,n}$ is equal to $1/(i + 1)$ times the last term of $S_{i+1,n+1}$; as, for instance, that

$$S_{1,n} = 1 + 2 + \cdots + n = \frac{n}{2}(n + 1),$$

that

$$S_{2,n} = 1\cdot 2 + 2\cdot 3 + \cdots + n(n + 1) = \tfrac{1}{3}(n - 1)n(n + 1),$$

etc.

## I. Solution by Howard C. Feemster, York College, Nebraska.

$$S_{1,n} = \frac{\underline{|1}}{\underline{|0}} + \frac{\underline{|2}}{\underline{|1}} + \cdots + \frac{\underline{|n}}{\underline{|n-1}} = 1 + 2 + \cdots + n = \frac{n}{2}(n+1);$$

$$S_{2,n} = \frac{\underline{|2}}{\underline{|0}} + \frac{\underline{|3}}{\underline{|1}} + \cdots + \frac{\underline{|n}}{\underline{|n-2}} = 1\cdot 2 + 2\cdot 3 + \cdots + (n-1)n = \tfrac{1}{3}n(n-1)(n+1),$$

for $n-1$ terms; and

$$S_{3,n} = \frac{\underline{|3}}{\underline{|0}} + \frac{\underline{|4}}{\underline{|1}} + \cdots + \frac{\underline{|n}}{\underline{|n-3}} = 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + \frac{(n-2)(n-1)n(n+1)}{4},$$

for $n-2$ terms.

Now assume

$$S_{i,n} = \frac{\underline{|i}}{\underline{|0}} + \frac{\underline{|i+1}}{\underline{|1}} + \frac{\underline{|1+3}}{\underline{|2}} + \cdots + \frac{\underline{|n}}{\underline{|n-i}} = 1\cdot 2\cdot 3\cdots i + 2\cdot 3\cdots(i+1)$$

$$(1) \qquad + 3\cdot 4\cdots(i+2) + \cdots + (n-i+1)(n-i+2)\cdots n$$

$$= \frac{(n-i+1)(n-i+2)\cdots(n)(n+1)}{i+1},$$

for $n-i+1$ terms.

But the formula is true for $n-i+1=1$ and $n-i+1=2$; for

$$\frac{\underline{|i}}{\underline{|0}} = \frac{1}{i+1}\cdot\frac{\underline{|i+1}}{\underline{|1}} \qquad \text{and} \qquad \frac{\underline{|i}}{\underline{|0}} + \frac{\underline{|i+1}}{\underline{|1}} = \frac{\underline{|i+2}}{i+1}.$$

If (1) is true

$$S_{i,n+1} = \frac{(n-i+1)(n-i+2)\cdots(n)(n+1)}{i+1} + (n-i+2)(n-i+3)\cdots n(n+1)$$

$$= \frac{(n-i+1)(n-i+2)\cdots n(n+1) + (i+1)(n-i+2)(n-i+3)\cdots(n)(n+1)}{i+1}$$

$$= \frac{(n-i+2)(n-i+3)\cdots(n)(n+1)(n+2)}{i+1},$$

which is of the same form as (1) when the number of terms is increased by unity, and hence the proposition is true in general.

## II. Solution by the Proposer.

Let $\phi(x) \equiv a^n + a^{n-1}x + a^{n-2}x^2 + \cdots + a^{n-i}x^i + \cdots + x^n$;
then

$$\frac{d^i\phi(x)}{dx^i} = \phi^{(i)}(x) \equiv i!a^{n-i} + \frac{(i+1)!}{1!}a^{n-i-1}x + \frac{(i+2)!}{2!}a^{n-i-2}x^2 + \cdots + \frac{n!}{(n-1)!}x^{n-i};$$

and

$$\phi(a) = na^n, \qquad \phi^{(i)}(a) = S_{i,n}a^{n-i}, \qquad \phi^{(n)}(a) = S_{n,n}a^\circ = n!.$$

Let $f(x) \equiv (x-a)\phi(x) \equiv x^{n+1} - a^{n+1}$; then $f(a) = 0$,

$$f'(x) \equiv (x-a)\phi'(x) + \phi(x); \qquad f'(a) = \phi(a) = na^n.$$

$$f''(x) \equiv (x-a)\phi''(x) + 2\phi'(x); \qquad f''(a) = 2\phi'(a) = 2S_{1,n}\cdot a^{n-1}.$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$f^{(i+1)}(x) \equiv (x-a)\phi^{(i+1)}(x) + (i+1)\phi^{(i)}(x); \quad f^{(i+1)}(a) = (i+1)\phi^{(i)}(a) = (i+1)S_{i,n}\cdot a^{n-i}.$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$f^{(n+1)}(a) = (n+1)\phi^{(n)}(a) = (n+1)!$$

Since $f(a+x) \equiv (a+x)^{n+1} - a^{n+1} \equiv \binom{n+1}{1}a^n x + \binom{n+1}{2}a^{n-1}x^2 + \cdots + \binom{n+1}{i+1}a^{n-i}x^{i+1} + \cdots + x^{n+1}$, and again,

$$f(a+x) \equiv f(a) + xf'(a) + \frac{x^2}{\underline{|2}}f''(a) + \cdots + \frac{x^{i+1}}{(i+1)!}f^{(i+1)}(a) + \cdots + \frac{x^{n+1}}{(n+1)!}(n+1)!,$$

we have, on comparison of coefficients,

$$\frac{1}{(i+1)!}f^{(i+1)}(a) = \binom{n+1}{i+1}a^{n-1} = \frac{(n+1)!}{(i+1)!(n-i)!}a^{n-1},$$

$$f^{(i+1)}(a) = (i+1)\phi^{(i)}(a) = (i+1)S_{i,\,n}\cdot a^{n-1} = \frac{(n+1)!}{(n-i)!}a^{n-1},$$

so that

$$S_{i,\,n} = \frac{1}{i+1}\cdot\frac{(n+1)!}{(n-i)!},$$

and $(n+1)!/(n-i)!$ is the last term of $S_{i+1,\,n+1} = \sum_{k=1}^{k=n-i+1} (i+k)!/(k-1)!$.

Also solved by HORACE OLSON.

**457. Proposed by FRANK IRWIN, University of California.**

If $a$ be any number prime to $m$ and $m/a$ be developed as a continued fraction,

$$\frac{m}{a} = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{k-1}} + \frac{1}{a_k},$$

with $a_1 \neq 0$, then there will exist a number $b$ such that

$$\frac{m}{b} = a_k + \frac{1}{a_{k-1}} + \cdots + \frac{1}{a_2} + \frac{1}{a_1}.$$

Show that $ab \equiv \pm 1 \pmod{m}$ and determine the sign.

## SOLUTION BY THE PROPOSER.

Write $m$ and $a$ as functions of the $a$'s: $m = [a_1, a_2, \cdots a_k]$, $a = [a_2, a_3, \cdots a_k]$, using the *Gaussische Klammer* notation, see BACHMANN, *Niedere Zahlentheorie*, vol. 1, p. 104; or, with a different symbol, $K(a_1, a_2, \cdots a_k)$, CHRYSTAL's *Algebra*, part II, 2d ed., p. 495.

Similarly, $b = [a_{k-1}, \cdots a_2, a_1]$, and if $p_{k-1}/q_{k-1}$ be the next to last convergent to $m/a$, $p_{k-1} = [a_1, a_2, \cdots a_{k-1}]$. But, by an elementary property of these expressions, $[a_1, a_2, \cdots a_{k-1}] = [a_{k-1}, \cdots a_2, a_1)$; that is, $p_{k-1} = b$.

Again, since $p_{k-1}/q_{k-1}$, and $m/a$ are successive convergents, $mq_{k-1} - ap_{k-1} = (-1)^k$; that is, $mq_{k-1} - ab = (-1)^k$, or $ab \equiv (-1)^{k-1} \pmod{m}$.

## GEOMETRY.

**483. Proposed by LAENAS G. WELD, Pullman, Illinois.**

A circle is inscribed in a triangle. In each of the three spandrils exterior to the circle another circle is inscribed; in the remaining spandrils three other circles; and so on ad infinitum. Show that the sum of the areas of these circles is given by the formula:

$$\Sigma = \frac{\pi}{4}\cdot\frac{\Delta^2}{S^2}\left[\frac{1}{\sin(A/2)} + \frac{1}{\sin(B/2)} + \frac{1}{\sin(C/2)} - 2 + \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right].$$

## SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let $A$, $B$, $C$ and $a$, $b$, $c$ denote the angles and sides opposite these angles of the given triangle. Since the center of the inscribed circle is at the intersection of the bisectors of the angles of the triangle, the centers of the circles inscribed in the spandrils are on these bisectors. Let $R$ be the radius of the inscribed circle and $R_1$, $R_2$, $\cdots$ $R_n$ the radius of the circles whose centers are $A_1$, $A_2$, $\cdots$ $A_n$, respectively.